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THE APERTURE ADMITTANCE OF A RECTANGULAR WAVEGUIDE RADIATING INTO A LOSSY HALF-SPACE

by

R.T. Compton, Jr. 30 Sep. 1963 51 p refs

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REPORT 1691-1

REPORT

by

THE OHIO STATE UNIVERSITY RESEARCH FOUNDATION
COLUMBUS 12, OHIO

Sponsor	National Aeronautics and Space Administration Washington 25, D.C.
Grant Number	NsG-448
Investigation of	Spacecraft Antenna Problems
Subject of Report	The Aperture Admittance of a Rectangular Waveguide Radiating into a Lossy Half-Space
Submitted by	R.T. Compton, Jr. Antenna Laboratory Department of Electrical Engineering
Date	30 September 1963

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THE APERTURE ADMITTANCE OF A RECTANGULAR WAVEGUIDE RADIATING INTO A LOSSY HALF-SPACE

INTRODUCTION

In this report the effective terminating admittance of a rectangular waveguide radiating into a lossy half-space is derived. The waveguide opens into the lossy region through an aperture in an infinite ground plane. The electric field in the aperture is assumed to have the form of the TE_{10} waveguide mode. The terminating admittance of the waveguide is then found by computing the complex power flowing through the aperture.

Curves of the terminating admittance have been plotted as a function of the complex propagation constant "k" in the lossy half-space for three common aperture sizes. Values of k corresponding to both positive and negative dielectric constants have been included, so that the curves will be useful in the design and interpretation of experiments for measuring plasma properties.

This material is intended to serve as a first approximation to the case of a thick plasma layer.

Although this general problem has been treated previously,* no information has been available for a half-space with an arbitrary permittivity and conductivity. In previous work, aperture admittance has been found as a function of aperture size or frequency, with the half-space assumed to be free-space.

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*See, for example, M.H. Cohen, T.H. Crowley, C.A. Levis, "The Aperture Admittance of a Rectangular Waveguide Radiating into Half-Space," The Ohio State University Antenna Laboratory Report 339-22, 14 November 1951. Also, L. Lewin, "Advanced Theory of Waveguide," Iliffe and Sons, Ltd., London, 1951, p. 121.

FORMULATION

Consider a rectangular waveguide which radiates through an infinite ground plane into a lossy half-space. The ground plane is assumed to be infinitely conducting. The waveguide opening in the ground plane has dimensions (a, b), as shown in Fig. 1.

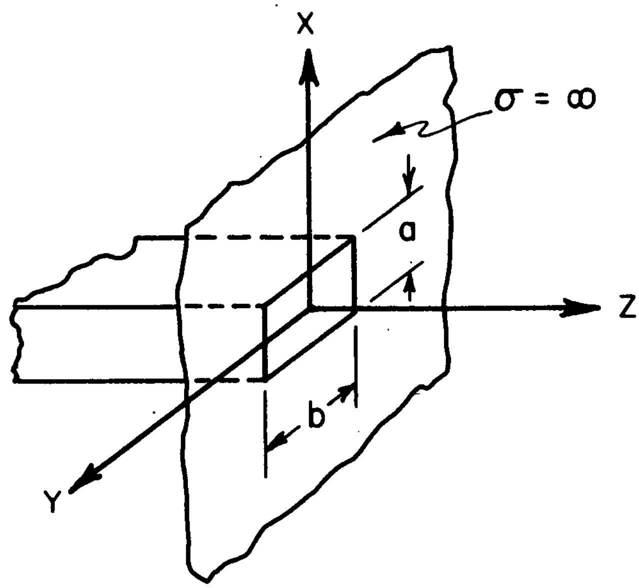

Fig. 1. Geometry of waveguide aperture.

The semi-infinite region $z > 0$ is assumed to be isotropic and homogeneous, and is characterized by a complex propagation constant

$$k = \left[\omega^2 \mu_o \epsilon \left(1 - i \frac{\sigma}{\omega\epsilon}\right)\right]^{\frac{1}{2}} \tag{1}$$

where k = complex propagation constant
ω = radian frequency
μ_o = permeability of free space
ϵ = permittivity of $z > 0$ region
σ = conductivity of $z > 0$ region.

The time convention $e^{+i\omega t}$ and rationalized MKS units will be used.

The electric field in the aperture is assumed to have the form of TE_{10} waveguide mode, with the electric field in the x-direction. Thus

$$E_x(x,y,o) = \begin{cases} \sqrt{\dfrac{2}{ab}} \cos \dfrac{\pi y}{b} : |x| \le \dfrac{a}{2}, \ |y| \le \dfrac{b}{2} \\ 0: \text{ elsewhere} . \end{cases} \tag{2}$$

The normalizing constant $\sqrt{2/ab}$ is included for the following reason. The transverse field components of the TE_{10} mode may be written

$$\bar{E}_t = V(z)\, \bar{e}_t(x,y) \tag{3}$$

$$\bar{H}_t = I(z)\, \bar{h}_t(x,y) \tag{4}$$

where $\bar{e}_t(x,y)$, $\bar{h}_t(x,y)$ are the vector mode functions* satisfying the normalization relations

$$\int_{x=-\frac{a}{2}}^{\frac{a}{2}} \int_{y=-\frac{b}{2}}^{\frac{b}{2}} |\bar{e}_t|^2 \, dx\, dy = \int_{x=-\frac{a}{2}}^{\frac{a}{2}} \int_{y=-\frac{b}{2}}^{\frac{b}{2}} |\bar{h}_t|^2 \, dx\, dy = 1,$$

and V(z), I(z) are the "transmission line" voltage and current. The constant $\sqrt{2/ab}$ included in (2) corresponds to an aperture field of unit voltage.

The aperture admittance Y will be found by computing the complex power **P** flowing through the aperture:

$$P = \frac{1}{2} \iint (\bar{E} \times \bar{H}^*) \cdot \hat{z}\, dx\, dy \tag{5}$$

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*Harrington, R. F., Time Harmonic Electromagnetic Fields, p. 383.

and then using the relation

$$(6) \qquad Y = \frac{2P^*}{|V|^2} = 2P^*$$

where V is the aperture "voltage" and the asterisk indicates the complex conjugate. The last equality in (6) is numerically correct because of our choice of unit aperture voltage.

With the aperture field as given in (2), the field is everywhere TE to the y-axis.* Hence the field may be represented by an electric vector potential

$$(7) \qquad \overline{F} = \hat{y}\psi$$

where ψ satisfies the wave equation

$$(8) \qquad \nabla^2\psi + k^2\psi = 0$$

with appropriate boundary conditions. The electric and magnetic fields are given by

$$(9) \qquad \overline{E} = -\nabla \times \overline{F}$$

$$(10) \qquad \overline{H} = \frac{1}{i\omega\mu_o}\left[\nabla(\nabla \cdot \overline{F}) + k^2\overline{F}\right] .$$

In particular

$$(11) \qquad E_x = \frac{\partial\psi}{\partial z}$$

---

*For a proof of this, see Appendix A.

(12) $$H_y = \frac{1}{i\omega\mu_o}\left[\frac{\partial^2\psi}{\partial y^2} + k^2\psi\right].$$

For ψ we choose a solution:

(13) $$\psi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x, k_y)\, e^{-ik_z z}\, e^{-ik_y y}\, e^{-ik_x x}\, dk_x\, dk_y$$

with

(14) $$k_z = \sqrt{k^2 - k_x^2 - k_y^2}$$

where the square root is chosen so that

(15) $$\mathrm{Re}(k_z) \geq 0$$

(16) $$\mathrm{Im}(k_z) \leq 0$$

corresponding to propagation in the +z-direction. Then from (11),

(17) $$E_x(x, y, z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -ik_z\, f(k_x, k_y)\, e^{-ik_z z}\, e^{-ik_y y}\, e^{-ik_x x}\, dk_x\, dk_y\,.$$

The inverse transform, evaluated at z = 0, gives

(18) $$-ik_z\, f(k_x, k_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x(x, y, o)\, e^{+ik_x x}\, e^{+ik_y y}\, dx\, dy.$$

Substituting for $E_x(x, y, o)$ from (2) results in

$$(19)\quad -ik_z f(k_x,k_y) = \frac{1}{(2\pi)^2}\sqrt{\frac{2}{ab}}\int_{y=-\frac{b}{2}}^{\frac{b}{2}}\int_{x=-\frac{a}{2}}^{\frac{a}{2}} \cos\frac{\pi y}{b}\, e^{+ik_y y}\, e^{+ik_x x}\, dx\, dy$$

$$= \frac{1}{4\pi^2}\sqrt{\frac{2}{ab}}\left[\frac{2}{k_x}\sin\left(\frac{k_x a}{2}\right)\right]\left[\frac{2\pi b\cos\left(\frac{k_y b}{2}\right)}{\pi^2 - k_y^2 b^2}\right] .$$

Hence

$$(20)\quad f(k_x,k_y) = \frac{i}{\pi k_x k_z}\sqrt{\frac{2b}{a}}\;\frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{\pi^2 - k_y^2 b^2}$$

and ψ in (13) is then

$$(21)\quad \psi = \sqrt{\frac{2b}{a}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{i}{\pi k_x k_z}\;\frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2 b^2)}\; e^{-ik_z z}\, e^{-ik_y y}\, e^{-ik_x x}\, dk_x\, dk_y .$$

Then from (11) and (12), E_x and H_y are found to be:

$$(22)\quad E_x = \sqrt{\frac{2b}{a}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\pi k_x}\;\frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2 b^2)}\; e^{-ik_z z}\, e^{-ik_x x}\, dk_x\, dk_y$$

$$(23)\quad H_y = \frac{1}{\omega\mu_o}\sqrt{\frac{2b}{a}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{(k^2-k_y^2)}{\pi k_x k_z}\;\frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2 b^2)}\; e^{-ik_z z}\, e^{-ik_y y}\, e^{-ik_x x}\, dk_x\, dk_y .$$

The complex power through the aperture is

$$(24)\quad P = \frac{1}{2}\int_{y=-\frac{b}{2}}^{\frac{b}{2}}\int_{x=-\frac{a}{2}}^{\frac{a}{2}} E_x(x,y,o)\, H_y^*(x,y,o)\, dx\, dy.$$

Since it is difficult to find $H_y(x,y,o)$ directly from (23), Parseval's theorem and the convolution theorem will be used to evaluate (24). For the Fourier transform pair

$$(25) \quad f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x,k_y)\, e^{-ik_x x}\, e^{-ik_y y}\, dk_x\, dk_y$$

$$(26) \quad F(k_x,k_y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) e^{+ik_x x}\, e^{+ik_y y}\, dx\, dy,$$

Parseval's theorem is:

$$(27) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x,y)\, f_2^*(x,y)\, dx\, dy = (2\pi)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(k_x,k_y) F_2^*(k_x,k_y)\, dk_x\, dk_y$$

and a special case of the convolution theorem is

$$(28) \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x,y) f_2(-x,-y) dx\, dy = (2\pi)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(k_x,k_y) F_2(k_x,k_y) dk_x\, dk_y.$$

$E_x(x,y,o)$ is zero outside the aperture. Hence the limits of integration in (24) may be extended to infinity. Then from Parseval's theorem and Eqs. (6), (22), (23), and (24), the admittance Y is found to be

$$(29) \quad Y = 2P^* = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_x^*(x,y,o)\, H_y(x,y,o)\, dx\, dy =$$

$$= (2\pi)^2 \frac{2b}{a\omega\mu_o} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(k^2 - k_y^2)}{\pi^2 k_x^2 k_z} \; \frac{\sin^2\left(\frac{k_x a}{2}\right)\cos^2\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2 b^2)^2}\, dk_x\, dk_y.$$

Next the terms of the integrand may be recombined as follows.

Let

$$(30) \quad F_1(k_x, k_y) = \frac{2b(k^2 - k_y^2)}{a\omega\mu_o \pi^2 k_x^2} \; \frac{\sin^2\left(\frac{k_x a}{2}\right)\cos^2\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2\, b^2)^2}$$

$$(31) \quad F_2(k_x, k_y) = \frac{1}{k_z} = \frac{1}{\sqrt{k^2 - k_x^2 - k_y^2}} \quad .$$

From Eq. (28),

$$(32) \quad Y = (2\pi)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(k_x, k_y) F_2(k_x, k_y) dk_x\, dk_y = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_1(x, y) f_2(-x, -y)\, dx\, dy$$

where $f_1(x, y)$, $f_2(x, y)$ are the transforms of $F_1(k_x, k_y)$.*

Now we proceed to find $f_1(x, y)$ and $f_2(x, y)$. Consider $f_1(x, y)$ first:

$$(33) \quad f_1(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_1(k_x, k_y)\, e^{-ik_x x}\, e^{-ik_y y}\, dk_x\, dk_y$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{2b(k^2 - k_y^2)}{a\omega\mu_o \pi^2 k_x^2} \; \frac{\sin^2\left(\frac{k_x a}{2}\right)\cos^2\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2\, b^2)^2}\, e^{-ik_x x}\, e^{-ik_y y}\, dk_x\, dk_y$$

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*This technique of rearranging the terms of the integrand and making repeated use of convolution theorems appears to be a useful trick for problems of this sort. By trying different combinations, one can find several equivalent integrals for P. The form chosen here seemed to be the most convenient for numerical evaluation.

$$(34)\quad = \frac{2b}{a\omega\mu_o\pi^2}\int_{-\infty}^{\infty}\frac{\sin^2\left(\frac{k_x a}{2}\right)}{k_x^2}\, e^{-ik_x x}\, dk_x \int_{-\infty}^{\infty}(k^2-k_y^2)\,\frac{\cos^2\left(\frac{k_y b}{2}\right)}{(\pi^2-k_y^2 b^2)^2}\, e^{-ik_y y}\, dk_y .$$

These integrals are easily done, and the result is :

$$(35)\quad f_1(x,y) = \frac{2b}{a\omega\mu_o\pi^2}\; g(x)\, h(y)$$

where

$$(36)\quad g(x) = \begin{cases} \frac{\pi}{2}(a - |x|) & : \quad |x| \le a \\ 0 & : \quad |x| > a \end{cases}$$

$$(37)\quad h(y) = \begin{cases} D_1(b-|y|)\cos\frac{\pi y}{b} + D_2 \sin\frac{\pi|y|}{b} & : \quad |y| \le b \\ 0 & : \quad |y| > b \end{cases}$$

and

$$(38)\quad D_1 = \frac{1}{b^2}\left[\frac{k^2}{4\pi} - \frac{\pi}{4b^2}\right]$$

$$(39)\quad D_2 = \frac{1}{b^2}\left[\frac{bk^2}{4\pi^2} + \frac{1}{4b}\right] = \frac{1}{\pi b}\left[\frac{k^2}{4\pi} + \frac{\pi}{4b^2}\right] .$$

Next, for $f_2(x,y)$, we have

$$(40)\quad f_2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F_2(k_x,k_y)\, e^{-ik_x x}\, e^{-ik_y y}\, dk_x\, dk_y$$

(41) $$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{e^{-ik_x x}\, e^{-ik_y y}}{\sqrt{k^2 - k_y^2 - k_x^2}}\, dk_x\, dk_y .$$

Doing the integration on k_x first, we find:*

(42) $$\int_{-\infty}^{\infty} \frac{e^{-ik_x x}\, dk_x}{\sqrt{k^2 - k_y^2 - k_x^2}} = +\,\pi\, H_0^{(2)}\left(|x|\sqrt{k^2 - k_y^2}\right).$$

The integration on k_y then yields

(43) $$f_2(x,y) = \int_{-\infty}^{\infty} \left[+\pi H_0^{(2)}\left(|x|\sqrt{k^2 - k_y^2}\right)\right] e^{-ik_y y}\, dk_y$$

(44) $$= 2\pi i \frac{e^{-ik\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}} .$$

The integral in (43) is known as Weyrich's integral.**

Thus, (32) gives

(45) $$Y = \int_{y=-b}^{b} \int_{x=-a}^{a} \frac{4bi}{a\omega\mu_0\pi}\, g(x)\, h(y)\, \frac{e^{-ik\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}}\, dx\, dy$$

and since $g(x) = g(-x)$ and $h(y) = h(-y)$, Y may be written

---

* Equation (42) is derived in Appendix B.

** W. Magnus and F. Oberhettinger, "Formulas and Theorems for the functions of Mathematical Physics," Chelsea Publishing Co., New York, 1954; p. 34.

$$(46) \quad Y = \frac{16\,b\,i}{a\omega\mu_o\pi} \int_{y=0}^{b} \int_{x=0}^{a} g(x)\,h(y)\, \frac{e^{-ik\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\, dx\, dy.$$

Finally, substituting for g(x), h(y),

$$(47) \quad Y = \frac{8\,b\,i}{a\omega\mu_o} \int_{y=0}^{b} \int_{x=0}^{a} (a-x) \left[D_1 (b-y) \cos \frac{\pi y}{b} + D_2 \sin \frac{\pi y}{b} \right] \frac{e^{-ik\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}\, dx\, dy.$$

It is convenient, for numerical evaluation of Y, to normalize Eq. (47) with respect to the free-space constants. Let k_o be the free-space propagation constant,

$$(48) \quad k_o = \omega\sqrt{\mu_o\,\epsilon_o} = \frac{2\pi}{\lambda_o}$$

(λ_o is the free-space wavelength), and let y_o be the free-space characteristic admittance,

$$(49) \quad y_o = \sqrt{\frac{\epsilon_o}{\mu_o}} \; .$$

Then (47) may be written

$$(50) \quad Y_n = \frac{Y}{y_o} = 8\,\frac{B}{A}\,i \int_{\eta=0}^{A} \int_{\xi=0}^{B} (A-\eta) \left[C_1 (B-\xi) \cos \frac{\pi\xi}{B} + C_2 \sin \frac{\pi\xi}{B} \right]$$

$$\cdot \; \frac{e^{-i\left(\frac{k}{k_o}\right)\sqrt{\eta^2+\xi^2}}}{\sqrt{\eta^2+\xi^2}}\, d\eta\, d\xi$$

where

$$(51) \quad A = k_o a$$

$$(52) \quad B = k_o b$$

$$(53) \quad C_1 = \frac{A_1}{k_o^4} = \frac{1}{4\pi B^2}\left[\left(\frac{k}{k_o}\right)^2 - \left(\frac{\pi}{B}\right)^2\right]$$

$$(54) \quad C_2 = \frac{A_2}{k_o^3} = \frac{1}{4\pi^2 B}\left[\left(\frac{k}{k_o}\right)^2 + \left(\frac{\pi}{B}\right)^2\right]$$

and Y_n is the normalized aperture admittance.

NUMERICAL RESULTS AND INTERPRETATION

Equation (50) has been evaluated in the university's Numerical Computation Laboratory with the IBM 1620 Digital Computer for three sizes of apertures:

(a) $A = \frac{\pi}{2}$, $B = \pi$ $\left(\frac{\lambda_o}{4} \text{ by } \frac{\lambda_o}{2}\right)$

(b) $A = \frac{3\pi}{4}$, $B = \frac{3\pi}{2}$ $\left(\frac{3\lambda_o}{8} \text{ by } \frac{3\lambda_o}{4}\right)$

(c) $A = \pi$, $B = 2\pi$ $\left(\frac{\lambda_o}{2} \text{ by } \lambda_o\right)$.

The computation was done by means of Simpson's rule, after an appropriate change of variables. The details of this calculation and the 1620 Computer program used are discussed in Appendix C.

The numerical results are shown in Figs. 2 through 6. The admittance Y_n is plotted in terms of normalized conductance G_n and normalized susceptance B_n:

$$(55) \quad Y_n = G_n + i\, B_n.$$

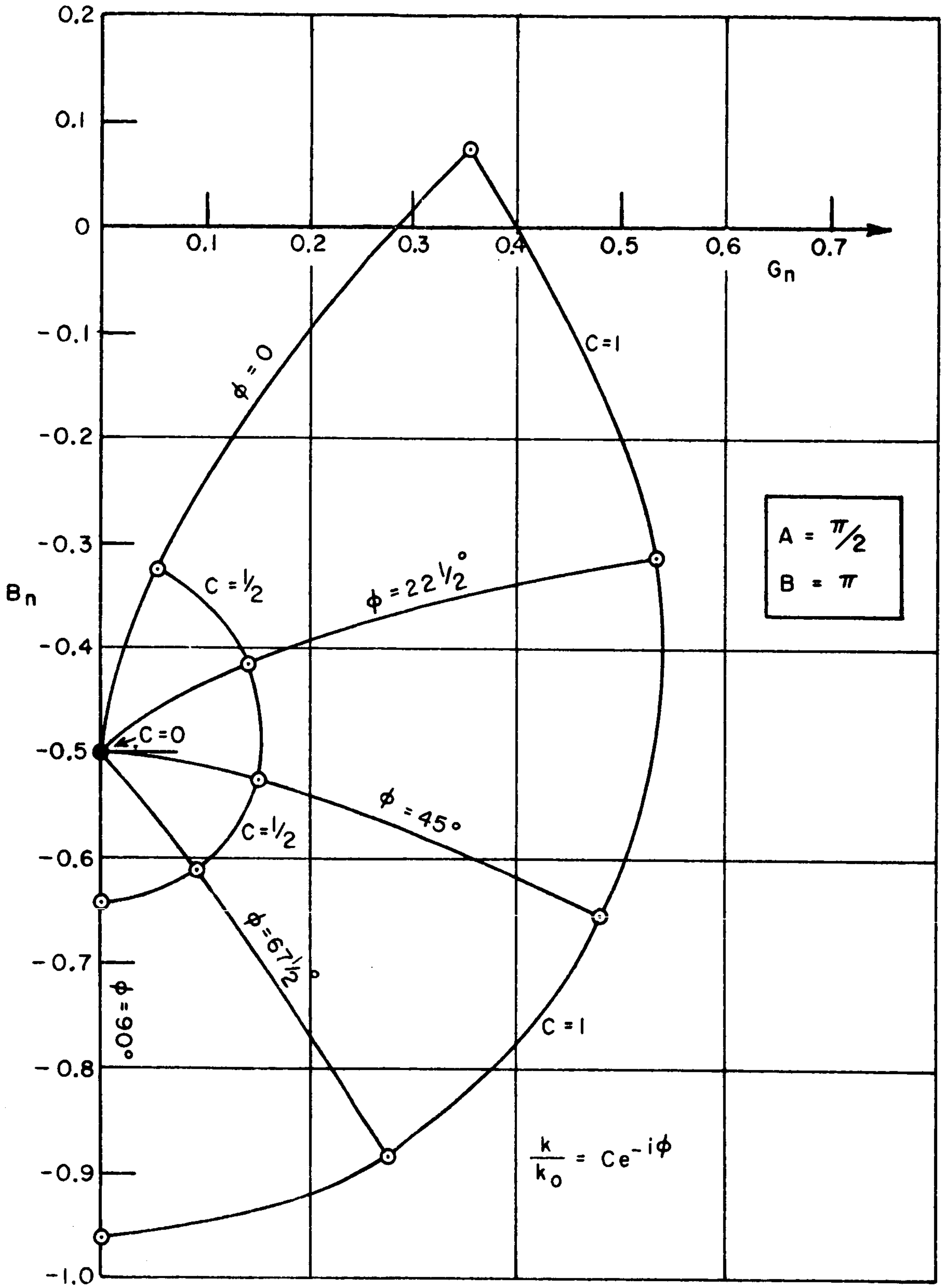

Fig. 2. Normalized aperture admittance.

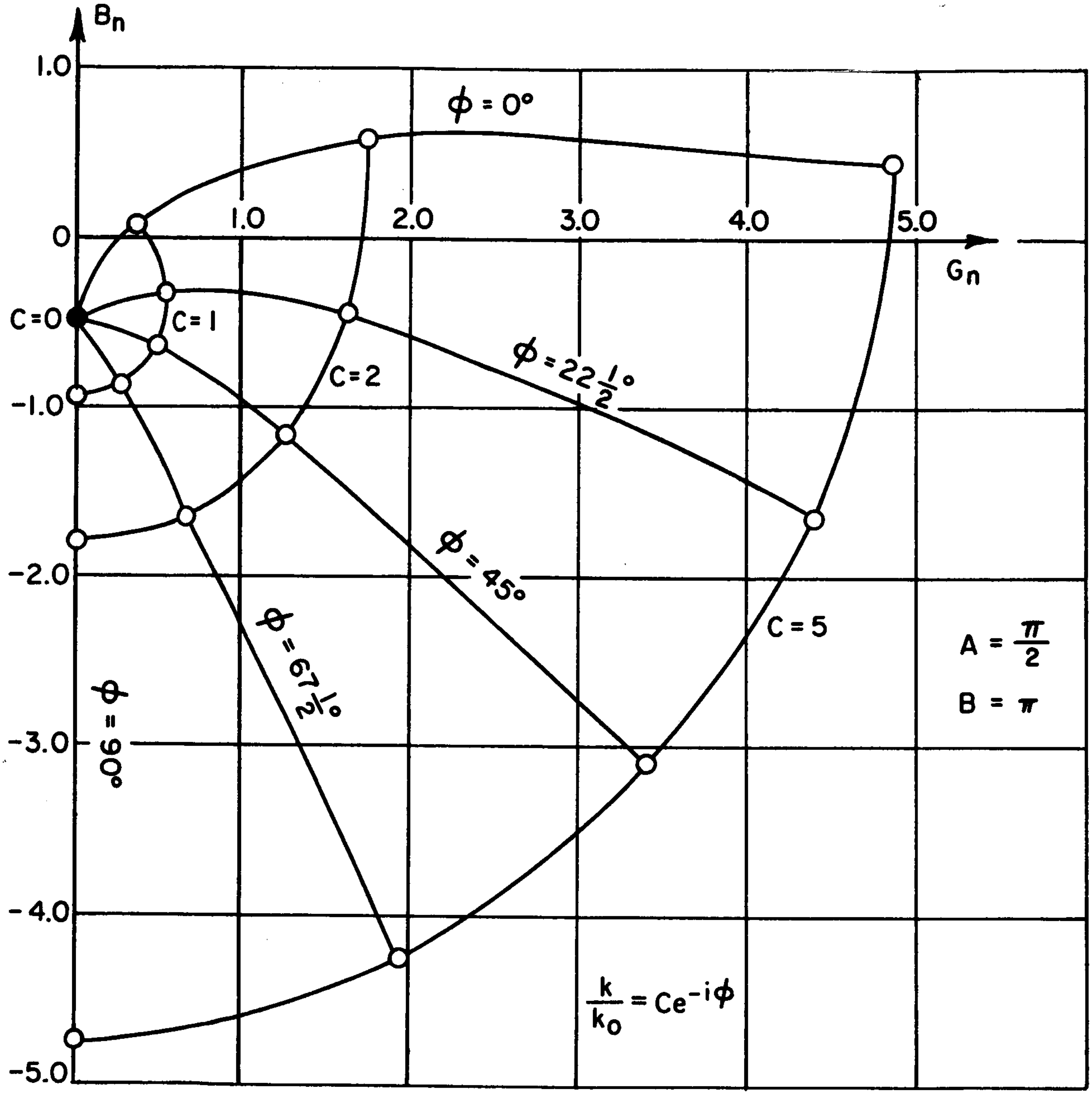

Fig. 3. Normalized aperture admittance.

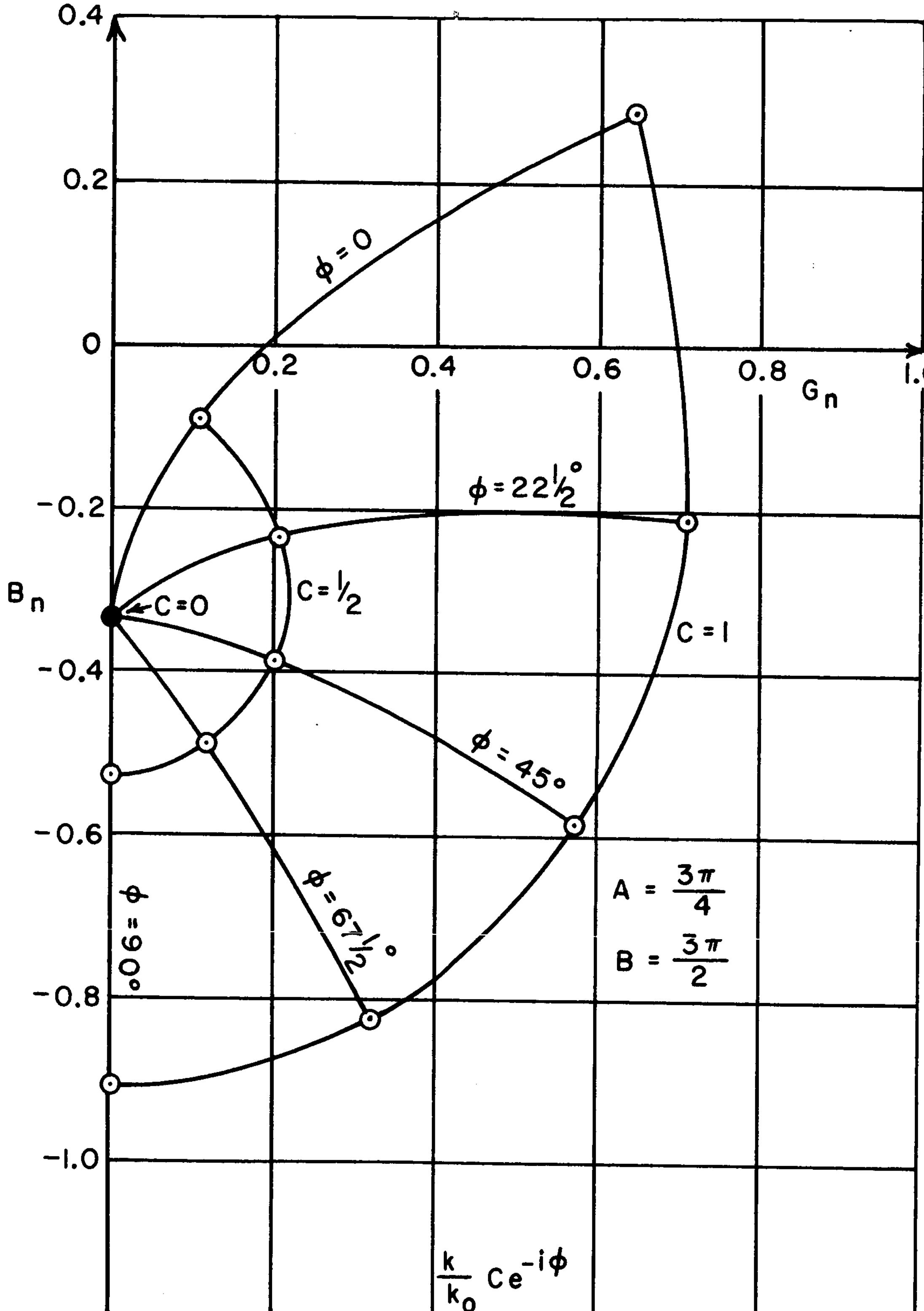

Fig. 4. Normalized aperture admittance.

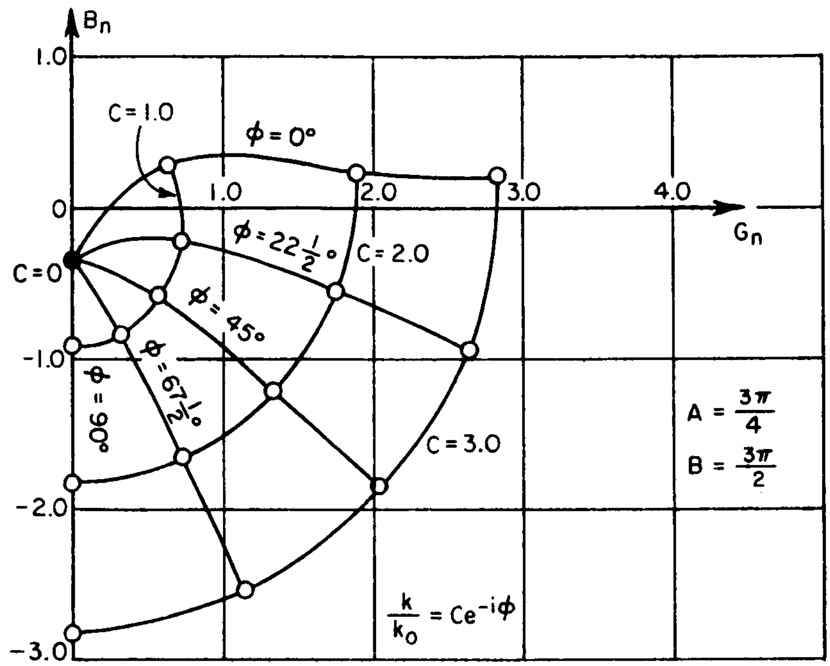

Fig. 5. Normalized aperture admittance.

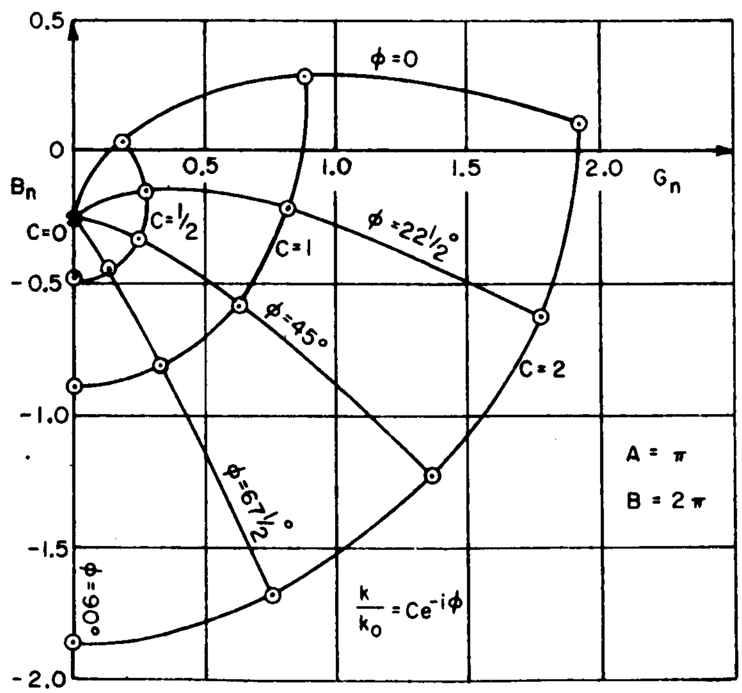

Fig. 6. Normalized aperture admittance.

With C and ϕ defined by

$$(56)\quad \frac{k}{k_o} = C\, e^{-i\phi},$$

the results are given for various values of C and ϕ.

Figures 2 and 3 show Y_n for the case $A = \frac{\pi}{2}$, $B = \pi$. Figure 2 gives the results for $0 \le C \le 1$ and $0 \le \phi \le 90°$, and Fig. 3 for $0 \le C \le 5$ and $0 \le \phi \le 90°$. Figures 4 and 5 show Y_n for the case $A = \frac{3\pi}{4}$, $B = \frac{3\pi}{2}$. In Fig. 4, the limits are $0 \le C \le 1$ and $0 \le \phi \le 90°$ and in Fig. 5, $0 \le C \le 3$. Figure 6 shows Y_n for $A = \pi$, $B = 2\pi$ and $0 \le C \le 2$, $0 \le \phi \le 90°$.

Finally, as a check on the numerical results, the integral for Y_n may be evaluated approximately for the case where k has a large (complex) value. In Eq. (50),

$$(50)\quad \frac{Y}{Y_o} = 8\,\frac{B}{A}\, i \int_{\eta=0}^{A}\int_{\xi=0}^{B} (A-\eta)\left[C_1(B-\xi)\cos\frac{\pi\xi}{B} + C_2 \sin\frac{\pi\xi}{B}\right]\frac{e^{-i\frac{k}{k_o}\sqrt{\eta^2+\xi^2}}}{\sqrt{\eta^2+\xi^2}}\, d\eta\, d\xi$$

the change of variables:

$$(57)\quad \eta = r\cos\theta$$

$$(58)\quad \xi = r\sin\theta$$

gives the substitution

$$(59)\quad \frac{e^{-i\left(\frac{k}{k_o}\right)\sqrt{\eta^2+\xi^2}}}{\sqrt{\eta^2+\xi^2}}\, d\eta\, d\xi = e^{-i\left(\frac{k}{k_o}\right)r}\, dr\, d\theta\,.$$

If (k/k_o) has a large (negative) imaginary part, the only contribution to the integral in (50) will occur in the vicinity of $r = 0$. In this region the other terms in the integrand may be approximated by

(60) $$(A-\eta) \simeq A$$

(61) $$C_1 (B-\xi) \cos \frac{\pi \xi}{B} \simeq C_1 B$$

(62) $$C_2 \sin \frac{\pi \xi}{B} \simeq 0 .$$

Also the range of integration on r may be extended to infinity with little change in the value of the integral. With these simplifications (50) becomes

(63) $$\frac{Y}{Y_o} = 8 \frac{B}{A} i \int_{r=0}^{\infty} \int_{\theta=0}^{\frac{\pi}{2}} A C_1 B e^{-i\left(\frac{k}{k_o}\right)r} dr\, d\theta$$

(64) $$= 4\pi C_1 B^2 \left(\frac{k}{k_o}\right) .$$

From (53), for large (k/k_o), C_1 becomes

(65) $$C_1 \simeq \frac{1}{4\pi B^2} \left(\frac{k}{k_o}\right)^2$$

so (64) yields

(66) $$\frac{Y}{Y_o} \simeq \frac{k}{k_o} ,$$

a surprisingly simple result. This behavior for large (k/k_o) is clearly indicated in Figs. 2 through 6.

For small (k/k_o) it is difficult to find a simple approximation for Y_n from Eq. (50). However, for the case where k/k_o is purely imaginary, it is easy to see that Eq. (50) gives a purely imaginary admittance, because C_1 and C_2 are real and the integrand has a real value.

The reason for this can be appreciated by examining Eq. (23) for the magnetic field. In the aperture, (23) gives:

$$(67)\quad H_y = \frac{1}{\omega\mu_o}\sqrt{\frac{2b}{a}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(k^2-k_y^2)}{\pi k_x k_z} \; \frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2 b^2)} \; e^{-ik_x x} \; e^{-ik_y y} \; dk_x \, dk_y .$$

This may be written

$$(68)\quad H_y = \frac{1}{\omega\mu_o}\sqrt{\frac{2b}{a}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(k_x, k_y)\, G_2(k_x, k_y)\, e^{-ik_x x} \; e^{-ik_y y} \; dk_x \, dk_y$$

where

$$(69)\quad G_1(k_x, k_y) = \frac{1}{k_z} = \frac{1}{\sqrt{k^2 - k_x^2 - k_y^2}}$$

$$(70)\quad G_2(k_x, k_y) = \frac{(k^2 - k_y^2)}{\pi k_x} \; \frac{\sin\left(\frac{k_x a}{2}\right)\cos\left(\frac{k_y b}{2}\right)}{(\pi^2 - k_y^2 b^2)} \quad .$$

Then by making use of the convolution theorem

$$(71)\quad \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(\eta, \xi)\, g_2(x-\eta, y-\xi)\, d\eta\, d\xi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(k_x, k_y)\, G_2(k_x, k_y) \cdot e^{-ik_x x} \; e^{-ik_y y} \; dk_x \, dk_y$$

where

$$(72)\quad g_1(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(k_x, k_y)\, e^{-ik_x x} \; e^{-ik_y y} \; dk_x \, dk_y$$

(73) $$g_2(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_2(k_x, k_y)\, e^{-ik_x x}\, e^{-ik_y y}\, dk_x\, dk_y$$

and the transform pairs given by Eq. (22) and Eqs. (40) through (44), H_y may be written

(74) $$H_y = \frac{1}{\omega\mu_o} \frac{i}{2\pi}\sqrt{\frac{2}{ab}}\left(k^2 + \frac{\partial^2}{\partial y^2}\right)\int_{\eta=-\frac{a}{2}}^{\frac{a}{2}} \int_{\xi=-\frac{b}{2}}^{\frac{b}{2}} \cos\frac{\pi}{b}(y-\xi) \cdot \frac{e^{-ik\sqrt{\eta^2+\xi^2}}}{\sqrt{\eta^2+\xi^2}}\, d\eta\, d\xi\ .$$

Now for the case where Re(k) = 0 , the integrand in (74) is real, k^2 is real, and hence H_y is purely imaginary. This means that the electric and magnetic fields in the aperture are in time quadrature. The complex power flow through the aperture, as given by Eq. (24), is therefore imaginary.

This situation is similar to the case of a large waveguide terminated by a small cutoff waveguide, as illustrated in Fig. 7.

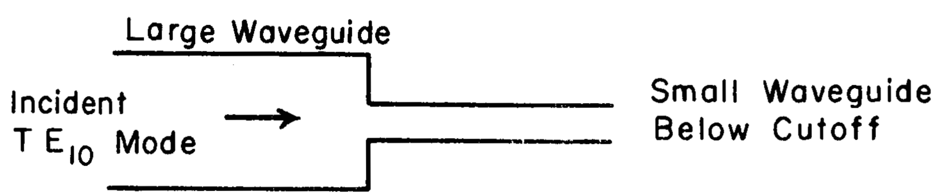

Fig. 7. Waveguide analogy.

In the small waveguide, which is cutoff, the electric and magnetic fields are in phase quadrature and the effective termination of the large guide is a pure susceptance.

It is interesting to note that for $k = 0$, the magnetic field is quasi-static. For a fixed aperture size and fixed frequency, the condition $k = 0$ corresponds to $\epsilon = 0$, $\sigma = 0$ in Eq. (1), which leads to a Laplace's equation for the magnetic field. (The case $k = 0$ can also be interpreted as the zero-frequency limit; but since the curves in Figs. 2 through 6 are plotted for constant $A = k_o a$ and $B = k_o b$, the physical aperture size must be considered as varying inversely with frequency in this case.)

CONCLUSIONS

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The terminating admittance of a rectangular waveguide radiating through a ground sheet into a lossy half-space has been found. Numerical results have been computed for three aperture sizes and are given in Figs. 2 through 6. It is noted that if the propagation constant k is imaginary, the aperture admittance is a pure inductive susceptance. Also, for large values of k, the normalized aperture admittance is approximately given by k/k_o, independent of the aperture dimensions.

These results should be useful for experimental measurements of the properties of a lossy medium (such as a plasma). From experimental values of aperture admittance, the propagation constant k may be found from Figs. 2 through 6. Then the permittivity and permeability of the medium can be found from k by Eq. (1). Author

ACKNOWLEDGEMENTS

The author is grateful to Professor Chen To Tai for numerous discussions and many helpful suggestions. Also, the assistance of Miss Mery Gong of the Numerical Computation Laboratory is appreciated.

APPENDIX A

The purpose of this section is to justify that the field generated by the aperture in Fig. 1 is TE to the y-axis.

The aperture field is assumed to have the form:

$$\text{(A-1)} \qquad E_x(x, y, o) = \begin{cases} \sqrt{\dfrac{2}{ab}} \cos \dfrac{\pi y}{b} : & |x| \leq \dfrac{a}{2}, \quad |y| \leq \dfrac{b}{2} \\ 0 & : \text{elsewhere.} \end{cases}$$

The fields in the region $z > 0$ generated by this $E_x(x, y, o)$ will be the same as those generated by a magnetic sheet current in the y-direction of the form

$$\text{(A-2)} \qquad K_y(x, y, o) = \begin{cases} -2\sqrt{\dfrac{2}{ab}} \cos \dfrac{\pi y}{b} : & |x| \leq \dfrac{a}{2}, \quad |y| \leq \dfrac{b}{2} \\ 0 & : \text{elsewhere} \end{cases}$$

which radiates in the center of an infinite lossy medium (i.e., with no ground plane). That this source is equivalent to the aperture field in (A-1) may be seen from the following discussion.

Suppose a source S, connected to the waveguide behind the ground plane, generates the fields $(\overline{E}, \overline{H})$ in the waveguide and in the lossy region, as shown in Fig. A-1.

Visualize a hypothetical surface "$\sum$" located a slight distance in front of the ground plane, as shown in Fig. A-2.

Now suppose electric and magnetic sheet currents of the form

$$\text{(A-3)} \qquad \begin{aligned} \overline{J}' &= \overline{H} \times \hat{n} \\ \overline{K}' &= \hat{n} \times \overline{E} \end{aligned}$$

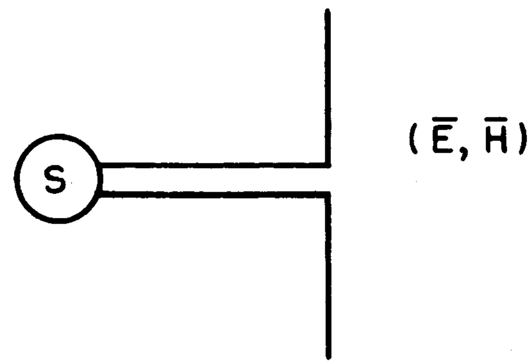

Fig. A-1. Source-excited aperture.

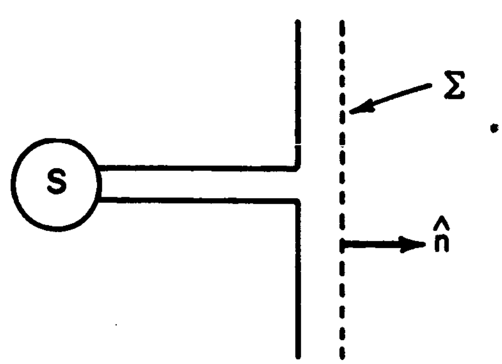

Fig. A-2. The surface Σ.

are placed on the surface $\sum$. With all sources, S, $\overline{J}'$, $\overline{K}'$, acting simultaneously, it may be seen that the fields to the left of $\sum$ will be the original $\overline{E}, \overline{H}$; the fields to the right of $\sum$ will be 0,0 (i.e., zero electric field and zero magnetic field). The reader may convince himself of this by noting that these fields satisfy Maxwell's equations in all regions and satisfy all boundary conditions.

Next suppose the sources $\overline{J}', \overline{K}'$ are allowed to radiate by themselves, with the source S turned off. From superposition, it is clear that $\overline{J}'$, $\overline{K}'$ will generate fields $-\overline{E}$, $-\overline{H}$ to the right of $\sum$ and 0,0 to the left of $\sum$. But since the fields resulting from $\overline{J}', \overline{K}'$ to the left of $\sum$ are zero, the ground sheet and waveguide structure could be removed without influencing the fields to the right of $\sum$. Thus we may consider $\overline{J}', \overline{K}'$ as radiating in an infinite medium with no ground plane.

Finally, we reverse the sign of the sources $\overline{J}', \overline{K}'$. I.e., let

(A-4)
$$\overline{J} = -\overline{J}' = \hat{n} \times \overline{H}$$
$$\overline{K} = -\overline{K}' = \overline{E} \times \hat{n}$$

flow on $\sum$. Then the fields to the right of $\sum$ will be $+\overline{E}, +\overline{H}$ and the fields to left are 0,0, as shown in Fig. A-3.

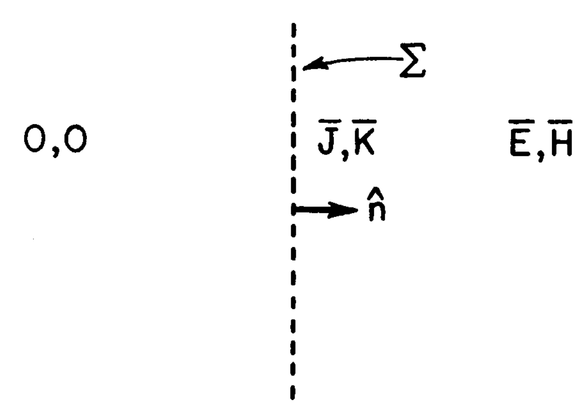

Fig. A-3. Equivalent sources.

Hence one possible set of equivalent sources for finding $\bar{E}, \bar{H}$ to the right of $\sum$ would be $\bar{J}, \bar{K}$ of Eq. (A-4) radiating in the infinite lossy medium.

However, a simpler equivalent source may be found as follows. Consider the effects of $\bar{J}$ and $\bar{K}$ separately. A single source $\bar{J}$ or $\bar{K}$ acting alone will generate fields on both sides of $\sum$. Let the fields resulting from $\bar{J}$ be $\bar{E}_J, \bar{H}_J$, and let those from $\bar{K}$ be $\bar{E}_K, \bar{H}_K$. For an electric source $\bar{J}$, the electric field $\bar{E}_J$ is symmetric and the magnetic field $\bar{H}_J$ is antisymmetric with respect to $\sum$.* For a magnetic source $\bar{K}$, the field $\bar{E}_K$ is antisymmetric and the field $\bar{H}_K$ is symmetric with respect to $\sum$. On the left of $\sum$, with both $\bar{J}$ and $\bar{K}$ operating,

$$
\text{(A-5)} \qquad \begin{aligned} \bar{E}_J + \bar{E}_K &= 0 \\ \bar{H}_J + \bar{H}_K &= 0 . \end{aligned}
$$

Therefore $\bar{E}_J = -\bar{E}_K$ and $\bar{H}_J = -\bar{H}_K$ on the left. But from the symmetry properties discussed above, it follows that on the right side of $\sum$,

$$
\text{(A-6)} \qquad \begin{aligned} \bar{E}_J &= +\bar{E}_K \\ \bar{H}_J &= +\bar{H}_K . \end{aligned}
$$

However, in this region

---

*If a field $\bar{A}$ is symmetric with respect to the $z = 0$ plane, it satisfies

$$
\begin{aligned} A_t(x, y, z) &= A_t(x, y, -z) \\ A_z(x, y, z) &= -A_z(x, y, -z) \end{aligned}
$$

where A_t is the component of $\bar{A}$ transverse to the $z = 0$ plane and A_z is the z-component (normal component) of $\bar{A}$. If the field $\bar{A}$ is antisymmetric with respect to the $z = 0$ plane, it safisfies

$$
\begin{aligned} A_t(x, y, z) &= -A_t(x, y, -z) \\ A_z(x, y, z) &= A_z(x, y, -z) . \end{aligned}
$$

$$(A-7)\qquad \begin{aligned}\bar{E}_J + \bar{E}_K &= \bar{E}\\ \bar{H}_J + \bar{H}_K &= \bar{H}\end{aligned}$$

so that on the right of $\sum$

$$(A-8)\qquad \begin{aligned}\bar{E}_J = \bar{E}_K &= \frac{\bar{E}}{2}\\ \bar{H}_J = \bar{H}_K &= \frac{\bar{H}}{2}\end{aligned} \quad .$$

Thus for the source $\bar{J}$ acting alone, the fields are as shown in Fig. A-4; and for K acting alone, the fields are as shown in Fig. A-5; where by $\pm\frac{\bar{E}}{2}$, $\pm\frac{\bar{H}}{2}$ on the left side of $\sum$ is meant the symmetric or antisymmetric image of the fields $\frac{\bar{E}}{2}, \frac{\bar{H}}{2}$ on the right side ($\bar{E}$ and $\bar{H}$ are the fields of the original problem to the right of $\sum$). By superimposing $\bar{J}$ and $\bar{K}$, it is now clear why the fields of both together are 0,0 on the left and $\bar{E}, \bar{H}$ on the right.

It also follows that a source $2\bar{K}$ acting alone or a source $2\bar{J}$ acting alone will produce $\bar{E}, \bar{H}$ to the right of $\sum$. Hence either one may be used alone as an equivalent source for the aperture field of Eq. (A-1). We choose to use $2\bar{K}$, of course, because the $\bar{E}$ field is assumed known.

Hence it has been justified that the source $2\bar{K} = 2\bar{E} \times \hat{n}$ in (A-2) is a suitable equivalent source for E_x in (A-1).

Next we show that K_y in (A-2) generates a field TE to the y-axis. To do this, we make use of the Carson form of the reciprocity theorem.*

Suppose a pair of sources $\bar{J}_a, \bar{K}_a$ radiate and generate the fields $\bar{E}_a, \bar{H}_a$, everywhere in space. Suppose also a second pair of sources $\bar{J}_b, \bar{K}_b$ (completely independent of the first pair) radiate, producing the fields E_b, H_b. Then the Carson reciprocity theorem states that

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*The fact that the field is TE to the y-axis could also be shown directly from the Dyadic Green's Function for the problem.

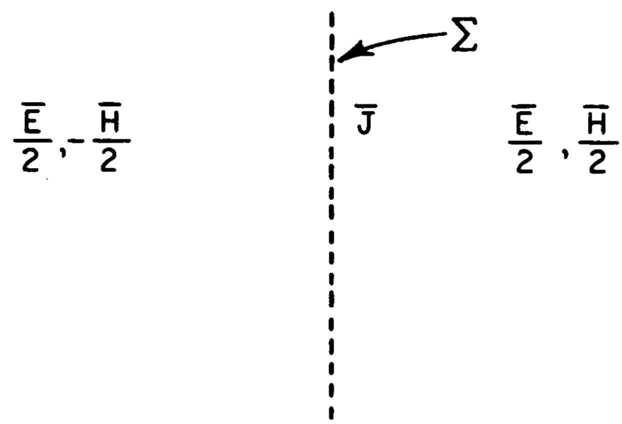

Fig. A-4. Equivalent sources.

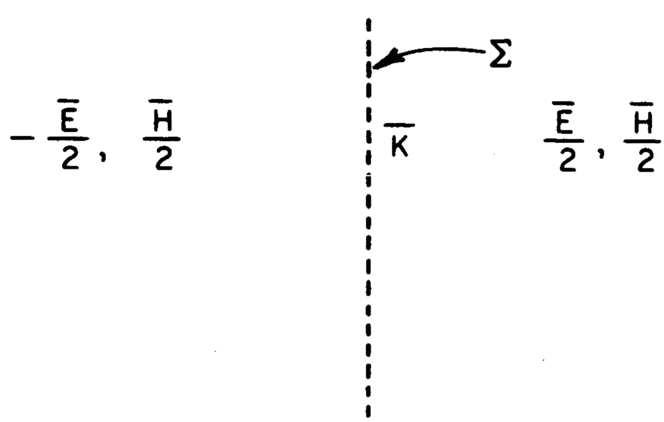

Fig. A-5. Equivalent sources.

$$(A-9) \qquad \iiint\limits_{V_a} (\bar{J}_a \cdot \bar{E}_b - \bar{K}_a \cdot \bar{H}_b) dv = \iiint\limits_{V_b} (\bar{J}_b \cdot \bar{E}_a - \bar{K}_b \cdot \bar{H}_a) dv$$

where V_a is the volume occupied by sources "a" and V_b is the volume occupied by sources "b".

In the present problem, let the "a" sources be the magnetic current given in (A-2), i.e.,

$$(A-10) \qquad \begin{aligned} \bar{J}_a &= 0 \\ \bar{K}_a &= \hat{y}\, K_y \ . \end{aligned}$$

Then $\bar{E}_a, \bar{H}_a$ will be the fields of the aperture problem. As the "b" source, let us choose a small electric dipole oriented in the +y-direction and located anywhere in the region $z > 0$. I.e.,

$$(A-11) \qquad \begin{aligned} \bar{J}_b &= \hat{y}\, \delta(x-x') \, \delta(y-y') \, \delta(z-z') \\ \bar{K}_b &= 0 \end{aligned}$$

where $\delta(x)$ is an impulse function at $x = 0$. The fields $\bar{E}_b, \bar{H}_b$ will then be the fields produced by this dipole. In particular, with the dipole oriented in the y-direction, it is known that the magnetic field $\bar{H}_b$ will have no y-component. Therefore, since $\bar{K}_a$ has only a y-component, there results

$$(A-12) \qquad \iiint\limits_{V_a} [\bar{J}_a \cdot \bar{E}_b - \bar{K}_a \cdot \bar{H}_b] dv \equiv 0$$

and hence from (A-9) it follows that

$$(A-13) \qquad \iiint\limits_{V_b} [\bar{J}_b \cdot \bar{E}_a - \bar{K}_b \cdot \bar{H}_o] dv = \iiint\limits_{V_b} (\bar{J}_b \cdot \bar{E}_a) dv = 0 .$$

But the integral in (A-13) is merely

$$\text{(A-14)} \qquad \iiint\limits_{V_b} \overline{J}_b \cdot \overline{E}_a \, dv = \iiint E_{ay}(x', y', z') \, \delta(x-x') \, \delta(y-y') \, \delta(z-z') \, dx' dy' dz'$$

$$= E_{ay}(x, y, z)$$

so that

$$\text{(A-15)} \qquad E_{ay}(x, y, z) = 0,$$

which we wished to prove.

APPENDIX B

EVALUATION OF EQUATION (42)

In Eq. (42), it is necessary to evaluate an integral of the form:

$$\text{(B-1)} \qquad I(\alpha, x) = \int_{-\infty}^{\infty} \frac{e^{-ik_x x}}{\sqrt{\alpha^2 - k_x^2}} \, dk_x$$

where α is complex and x is real. Define

$$\text{(B-2)} \qquad \alpha = \alpha' - i\alpha'' .$$

To evaluate (42), the substitution

$$\text{(B-3)} \qquad \alpha^2 \doteq k^2 - k_y^2$$

will be made. Since k_y is real and

$$\text{(B-4)} \qquad \operatorname{Im}(k^2) \leq 0,$$

it is noted that for this problem α can be restricted to the quadrant

$$\text{(B-5)} \qquad \alpha' \geq 0$$

$$\text{(B-6)} \qquad \alpha'' \geq 0 .$$

Consider first the case where $x > 0$. To integrate (B-1), let

$$\text{(B-7)} \qquad k_x = -\alpha \cos \theta$$

where

$$\text{(B-8)} \qquad \theta = \theta' + i\theta'' .$$

The path of integration in (B-1) is given by

(B-9) $$\mathrm{Im}(k_x) = 0$$

or

(B-10) $$\mathrm{Im}(-\alpha \cos\theta) = \mathrm{Im}[-(\alpha' - i\alpha'')\cos(\theta' + i\theta'')]$$
$$= \alpha' \sin\theta' \sinh\theta'' + \alpha'' \cos\theta' \cosh\theta'' = 0 .$$

Hence the equation of the contour in the θ-plane is:

(B-11) $$\tan\theta' \tanh\theta'' = -\frac{\alpha''}{\alpha'} .$$

Since

(B-12) $$\lim_{\theta'' \to +\infty} \tanh\theta'' = +1$$

(B-13) $$\lim_{\theta'' \to -\infty} \tanh\theta'' = -1$$

as θ'' goes to $+\infty$, $\tan\theta'$ approaches $-\frac{\alpha''}{\alpha'}$ and as θ'' goes to $-\infty$, $\tan\theta'$ approaches $+\frac{\alpha''}{\alpha'}$. If the path nearest the origin of the θ-plane is chosen, the contour is as shown in Fig. B-1, where

(B-14) $$\theta_1' = \tan^{-1}\left(+\frac{\alpha''}{\alpha'}\right)$$

(B-15) $$\theta_2' = \tan^{-1}\left(-\frac{\alpha''}{\alpha'}\right) .$$

Also since

(B-16) $$\mathrm{Re}(k_x) = \mathrm{Re}(-\alpha \cos\theta) = -\alpha' \cos\theta' \cosh\theta'' + \alpha'' \sin\theta' \sinh\theta'' ,$$

and for the region $\theta_1' < \theta' < \frac{\pi}{2}$, $\cos\theta' > 0$, $\sin\theta' > 0$, $\sinh\theta'' < 0$, and $\cosh\theta'' > 0$, it follows that the point $\theta' = \theta_1'$, $\theta'' = -\infty$ corresponds to $k_x = -\infty$. Similarly, the point $\theta' = \theta_2'$, $\theta'' = +\infty$ corresponds to $k_x = +\infty$. Hence the direction of integration is as indicated by the arrows in Fig. B-1.

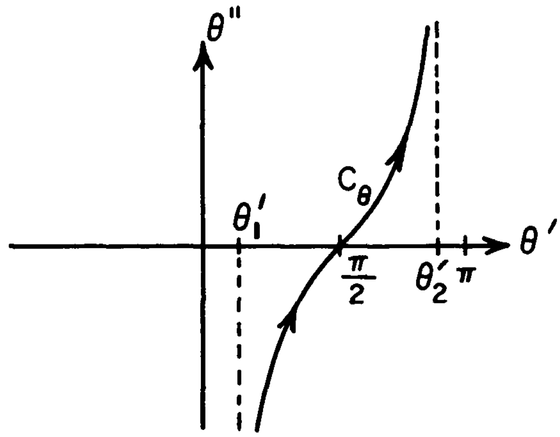

Fig. B-1. Contour of integration.

With the path of integration located, the correct sign of the square root in Eq. (B-1) can be determined. From Eq. (14) (of the main report),

$$k_z = \sqrt{k^2 - k_y^2 - k_x^2} = \sqrt{\alpha^2 - k_x^2} \tag{B-17}$$

and from Eqs. (15) and (16), the root should be chosen so that

$$\mathrm{Re}(k_z) \geq 0 \tag{B-18}$$

$$\mathrm{Im}(k_z) \leq 0 \ . \tag{B-19}$$

The correct choice is

$$\sqrt{\alpha^2 - k_x^2} = +\alpha \sin\theta \ . \tag{B-20}$$

That this is correct may be seen as follows. First, if (B-20) is used, (B-21) $\mathrm{Re}(k_z) = \mathrm{Re}(\alpha \sin\theta) = \alpha' \sin\theta' \cosh\theta'' + \alpha'' \cos\theta' \sinh\theta''$

and

(B-22) $\mathrm{Im}(k_z) = \mathrm{Im}(\alpha \sin\theta) = \alpha' \cos\theta' \sinh\theta'' - \alpha'' \sin\theta' \cosh\theta''$.

For the region $\theta_1' < \theta' < \frac{\pi}{2}$, $\sin\theta' > 0$, $\cos\theta' > 0$, $\sinh\theta'' < 0$, $\cosh\theta'' > 0$, so

(B-23) $\mathrm{Im}(k_z) = \alpha' \cos\theta' \sinh\theta'' - \alpha' \sin\theta' \cosh\theta'' < 0$.

Also for $\theta_1' < \theta' < \frac{\pi}{2}$,

(B-24) $\tan\theta' = \dfrac{\sin\theta'}{\cos\theta'} > \dfrac{\alpha''}{\alpha'}$

or

(B-25) $\alpha' \sin\theta' > \alpha'' \cos\theta'$,

and because $\theta'' < 0$ in this region,

(B-26) $\cosh\theta'' > -\sinh\theta''$,

so that from (B-25) and (B-26),

(B-27) $\alpha' \sin\theta' \cosh\theta'' > \alpha'' \cos\theta'(-\sinh\theta'')$

and therefore

(B-28) $\mathrm{Re}(k_z) = \alpha' \sin\theta' \cosh\theta'' + \alpha'' \cos\theta' \sinh\theta'' > 0$.

For the region $\frac{\pi}{2} < \theta' < \theta_2'$ function signs are $\sin\theta' > 0$, $\cos\theta' < 0$, $\sinh\theta'' > 0$, and $\cosh\theta'' > 0$, so that

(B-29) $\mathrm{Im}(k_z) = \alpha' \cos\theta' \sinh\theta'' - \alpha'' \sin\theta' \cosh\theta'' < 0$.

Also for $\frac{\pi}{2} < \theta' < \theta_2'$,

(B-30) $$\tan\theta' = \frac{\sin\theta'}{\cos\theta'} < -\frac{\alpha''}{\alpha'}$$

or

(B-31) $$\alpha' \sin\theta' > \alpha'' (-\cos\theta)$$

and because $\theta'' > 0$ in this region,

(B-32) $$\cosh\theta'' > \sinh\theta''$$

so that from (B-31) and (B-32)

(B-33) $$\alpha' \sin\theta' \cosh\theta'' > \alpha'' (-\cos\theta') \sinh\theta'' .$$

Hence

(B-34) $$\mathrm{Re}(k_z) = \alpha' \sin\theta' \cosh\theta'' + \alpha'' \cos\theta' \sinh\theta'' > 0 .$$

Thus it is seen that Eqs. (B-18) and (B-19) are satisfied for the choice of sign in (B-20).

Therefore with the substitution (B-7), (B-1) becomes

(B-35) $$I(\alpha, x) = \int_{C_\theta} e^{+i\alpha x \cos\theta}\, d\theta$$

where C_θ is the contour in Fig. B-1.

For the case where $x < 0$, Eq. (B-1) may be written

(B-36) $$I(\alpha, x) = \int_{-\infty}^{\infty} \frac{e^{+ik_x |x|}}{\sqrt{\alpha^2 - k_x^2}}\, dk_x .$$

For this case the substitution

(B-37) $\quad k_x = +\alpha \cos\theta$

is appropriate. The path of integration is found from

(B-38) $\quad \mathrm{Im}(k_x) = 0$

as above. Since the change of sign between (B-7) and (B-38) does not change the result (B-11), the contour is the same as for $x > 0$. The only difference is that the direction of integration is reversed from C_θ. For the radical in (B-36), the correct sign is

(B-39) $\quad \sqrt{\alpha^2 - k_x^2} = +\alpha \sin\theta$

because, as has been shown above, this sign satisfies (B-18) and (B-19) on C_θ.

Substituting (B-39) and (B-38) in (B-36) gives

(B-40) $$I(\alpha, x) = -\int_{-C_\theta} e^{+i|x|\alpha\cos\theta}\, d\theta$$

where "$-C_\theta$" means "along the path C_θ but in the opposite direction to C_θ".

Finally, by changing the direction of integration to $+C_\theta$ and dropping the minus sign, (B-40) may be written

(B-41) $$I(\alpha, x) = \int_{C_\theta} e^{+i|x|\alpha\cos\theta}\, d\theta .$$

Equation (B-35), which holds only for $x > 0$, is seen to be identical with (B-41) for $x > 0$, and therefore (B-41) is correct for either $x > 0$ or $x < 0$.

Finally, (B-41) may be evaluated with the aid of Sommerfeld's contour integral for the Hankel Function*

$$H_o^{(2)}(\rho) = \frac{1}{\pi} \int_{C_H} e^{i\rho \cos \theta} \, d\theta \tag{B-42}$$

where the contour C_H is shown in Fig. B-2 along with C_θ.

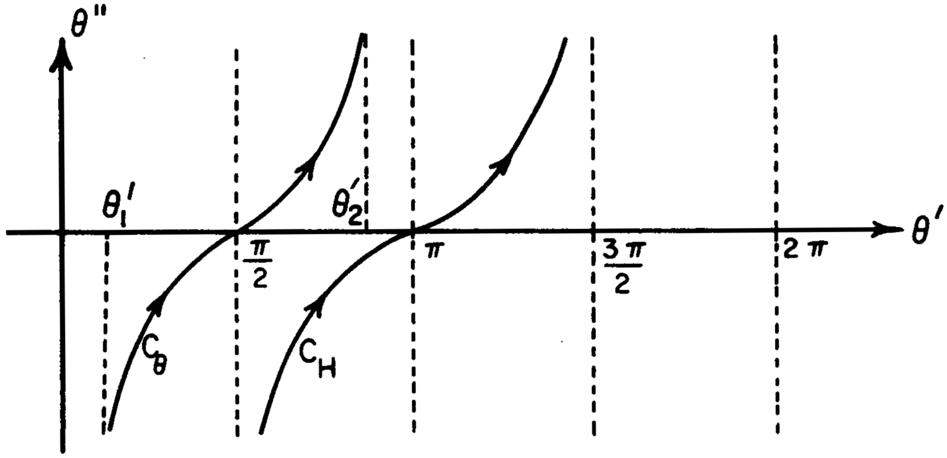

Fig. B-2. Hankel function contour.

Since the integrand in (B-41) has no singularities in the region between C_θ and C_H in the θ-plane, it is plausible that the integral (B-41) will have the same value whether the contour is taken along C_θ or C_H. To prove that this is the case, it is necessary first to locate those regions in the complex θ-plane for which $e^{+i|x|\alpha \cos\theta}$ approaches zero as θ'' goes to $\pm\infty$. (For any integral of the type (B-41), it is clear that the path of integration must go to infinity in these regions if the integral is to converge.)

First, from the relation

$$e^{i|x|\alpha \cos \theta} = e^{i|x| \mathrm{Re}(\alpha \cos\theta)} \, e^{-|x| \mathrm{Im}(\alpha \cos\theta)} \tag{B-43}$$

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*"Partial Differential Equations in Physics," A. Sommerfeld, Academic Press, Inc., 1949, Chap. IV.

it may be seen that the dividing lines between the regions of convergence and the regions of divergence satisfy the equation

(B-44) $$\mathrm{Im}(\alpha \cos \theta) = 0 .$$

Hence from (B-10) and (B-11), the dividing lines satisfy:

(B-45) $$\tan \theta' \tanh \theta'' = -\frac{\alpha''}{\alpha'} .$$

Three such lines are shown in Fig. B-3. (The line separating region B

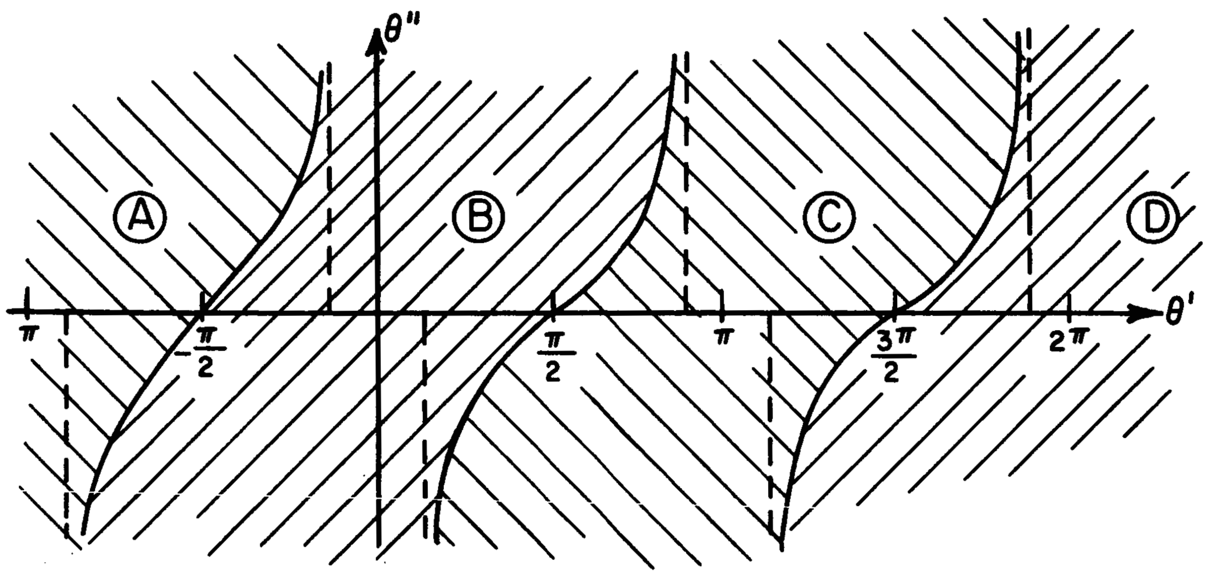

Fig. B-3. Regions of convergence and divergence.

from region C is the path C_θ.) Now since

(B-45) $$\alpha = \alpha' - i\alpha'' = |\alpha| e^{-i\theta_1'} = |\alpha| \cos \theta_1' - i|\alpha| \sin \theta_1' ,$$

it follows that

$$\text{(B-46)} \qquad \mathrm{Re}[i|x|\alpha\cos\theta] = \mathrm{Re}[i|x|(|\alpha|\cos\theta_1' - i|\alpha|\sin\theta_1') \cdot (\cos\theta'\cosh\theta'' - i\sin\theta'\sinh\theta'')]$$

$$= \frac{|x||\alpha|}{2}[\sin(\theta'+\theta_1')e^{\theta''} - \sin(\theta'-\theta_1')e^{-\theta''}]$$

and it may be seen that for points in regions B and D

$$\text{(B-47)} \qquad \lim_{\theta''\to+\infty} e^{+i|x|\alpha\cos\theta} = \lim_{\theta''\to-\infty} e^{+i|x|\alpha\cos\theta} = +\infty ,$$

and for points in A and C

$$\text{(B-48)} \qquad \lim_{\theta''\to+\infty} e^{+i|x|\alpha\cos\theta} = \lim_{\theta''\to-\infty} e^{+i|x|\alpha\cos\theta} = 0 .$$

Thus integrals with contours going to infinity in regions B and D diverge, those with contours going to infinity in regions A and C converge. The Hankel Function contour in Fig. B-2 of course gives a convergent integral, because it is completely in region C . The contour C_θ of Eq. (B-41) lies on the boundary between region B and region C .

Now the fact that

$$\text{(B-49)} \qquad I(\alpha,x) = \int_{C_\theta} e^{+i|x|\alpha\cos\theta}\, d\theta = \int_{C_H} e^{+i|x|\alpha\cos\theta}\, d\theta = \pi H_0^{(2)}(\alpha|x|)$$

may be proved as follows. Consider the closed contour of Fig. B-4. C_1 is the portion of C_θ lying between $-A \leq \theta'' \leq A$, C_2 is a horizontal contour at $\theta'' = +A$, C_3 is the portion of the C_H contour between $-A \leq \theta'' \leq A$, and C_4 is a horizontal contour at $\theta'' = -A$. Since

$$\text{(B-50)} \qquad f(\theta) = e^{+i|x|\alpha\cos\theta}$$

is an analytic function of θ for all finite values of θ, from the Cauchy integral theorem it follows that

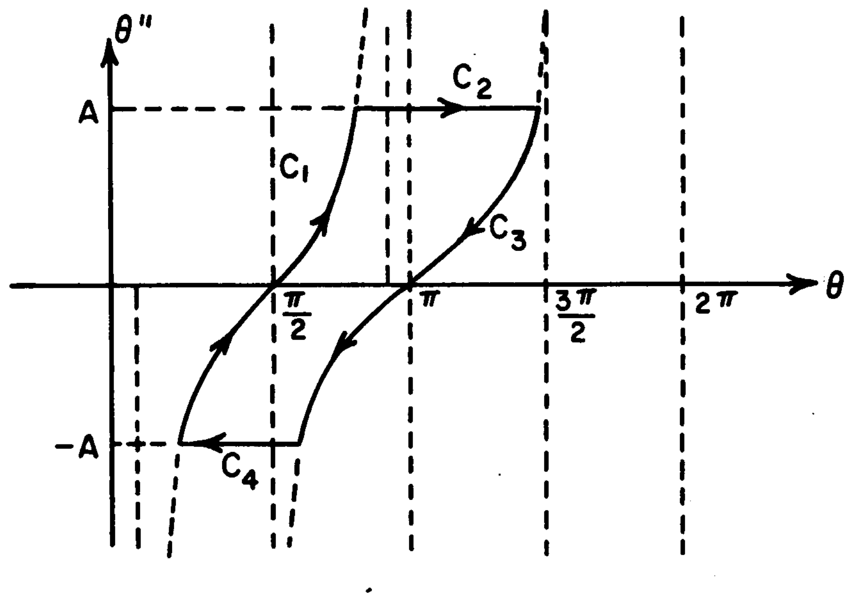

Fig. B-4. Closed contour of integration.

(B-51) $$\int_{C_1+C_2+C_3+C_4} f(\theta)d\theta = 0 .$$

We will show below that

$$\lim_{A\to\infty} \int_{C_2} f(\theta)d\theta = \lim_{A\to\infty} \int_{C_4} f(\theta)d\theta = 0$$

which will give us the desired relation between the integral on C_θ and on C_H.

Consider the integral

$$\int_{C_2} f(\theta)d\theta .$$

(B-52) $$0 \leq \left| \int_{C_2} e^{+i|x|\alpha \cos \theta} d\theta \right| \leq \int_{C_2} \left| e^{i|x|\alpha \cos \theta} \right| d\theta = \int_{C_2} e^{\mathrm{Re}[i|x|\alpha \cos \theta]} d\theta$$

$$= \int_{C_2} e^{|x|\alpha' \sin \theta' \sinh A + |x|\alpha'' \cos \theta' \cosh A} d\theta' .$$

The left end-point of C_2 is given by (from Eq. (B-11)):

(B-53) $$\theta' = \theta_o' = \tan^{-1}\left[\frac{-\alpha''/\alpha'}{\tanh A}\right]$$

and the right end-point of C_2 is less than $\frac{3\pi}{2}$, so

(B-54) $$\left| \int_{C_2} e^{+i|x|\alpha \cos \theta} d\theta \right| \leq \int_{\theta'=\theta_o}^{\frac{3\pi}{2}} e^{|x|\alpha' \sin \theta' \sinh A + |x|\alpha'' \cos \theta' \cosh A} d\theta'$$

(B-55) $$= \int_{\theta'=\theta_o}^{\frac{3\pi}{2}} e^{|x|\alpha'' \cos \theta' \cosh A \left[1 + \frac{\alpha'}{\alpha''} \tan \theta' \tanh A\right]} d\theta' .$$

On the whole path C_2, $\cos \theta'$ is negative and there is a constant $B > 0$ such that

(B-56) $$\cos \theta' < -B < 0 .$$

Consider the function

$$\left[1 + \frac{\alpha'}{\alpha''} \tan \theta' \tanh A\right].$$

For $\theta' = \theta_o$, it is zero. Since for every θ'

(B-57) $$\frac{d}{d\theta'}[\tan\theta'] \geq 1,$$

the inequality

(B-58) $$0 \leq \left[1 + \frac{\alpha'}{\alpha''}(\tanh A)(\theta' - \theta_o)\right] \leq \left[1 + \frac{\alpha'}{\alpha''}\tan\theta' \tanh A\right]$$

holds for $\theta_o \leq \theta' \leq \frac{3\pi}{2}$. Therefore the integral in (B-55) satisfies the inequality:

(B-59) $$\int_{\theta'=\theta_o}^{\frac{3\pi}{2}} e^{|x|\alpha''\cos\theta'\cosh A\left[1 + \frac{\alpha'}{\alpha''}\tan\theta'\tanh A\right]} d\theta'$$

$$\leq \int_{\theta'=\theta_o}^{\frac{3\pi}{2}} e^{-|x|\alpha''B\cosh A\left[1 + \frac{\alpha'}{\alpha''}(\tanh A)(\theta' - \theta_o)\right]} d\theta'$$

$$= \frac{e^{-|x|\alpha' B\theta_o \sinh A} - e^{-|x|\alpha''B\cosh A - |x|\alpha' B\left(\frac{3\pi}{2} - \theta_o\right)\sinh A}}{|x|\alpha' B \sinh A} = h(A).$$

Combining (B-54) through (B-59) and taking the limit as A goes to $+\infty$ gives the result:

$$0 \leq \lim_{A\to\infty}\left|\int_{C_2} e^{+i|x|\alpha\cos\theta} d\theta\right| \leq \lim_{A\to\infty} h(A) = 0$$

or

(B-60) $$\lim_{A\to\infty}\int_{C_2} e^{+i|x|\alpha\cos\theta} d\theta = 0 .$$

In an analogous manner, it is simple to show that for the integral along C_4:

$$\text{(B-61)} \qquad \lim_{A \to \infty} \int_{C_4} e^{+i|x|\alpha \cos \theta} d\theta = 0 .$$

As A goes to infinity, C_1 becomes C_θ and C_3 becomes $-C_H$. Therefore from (B-51)

$$\text{(B-62)} \qquad \int_{C_\theta} e^{+i|x|\alpha \cos \theta} d\theta = -\int_{-C_H} e^{+i|x|\alpha \cos \theta} d\theta = \int_{C_H} e^{+i|x|\alpha \cos \theta} d\theta$$

or

$$\text{(B-63)} \qquad \int_{C_\theta} e^{+i|x|\alpha \cos \theta} d\theta = \pi H_o^{(2)} (\alpha |x|)$$

which is the desired result.

APPENDIX C

In this appendix the method used for evaluating the aperture admittance numerically is discussed.

From (50), the normalized aperture admittance is given by

$$Y_n = \frac{Y}{Y_o} = 8\,\frac{B}{A}\,i \int_{\eta=0}^{A} \int_{\xi=0}^{B} (A-\eta)\left[C_1\,(B-\xi)\cos\frac{\pi\xi}{B} + C_2 \sin\frac{\pi\xi}{B}\right] \frac{e^{-i\left(\frac{k}{k_o}\right)\sqrt{\eta^2+\xi^2}}}{\sqrt{\eta^2+\xi^2}}\, d\eta\, d\xi \tag{C-1}$$

where

$$C_1 = \frac{1}{4\pi B^2}\left[\left(\frac{k}{k_o}\right)^2 - \left(\frac{\pi}{B}\right)^2\right] \tag{C-2}$$

$$C_2 = \frac{1}{4\pi^2 B}\left[\left(\frac{k}{k_o}\right)^2 + \left(\frac{\pi}{B}\right)^2\right]. \tag{C-3}$$

Making the change of variables

$$\eta = R\cos\theta \tag{C-4}$$

$$\xi = R\sin\theta \tag{C-5}$$

gives:

(C-6) $$Y_n = 8\frac{B}{A} i \int_{\theta=0}^{\theta_o} \int_{R=0}^{\frac{A}{\cos\theta}} (A - R\cos\theta) \Big[C_1 (B - R\sin\theta) \cos\frac{\pi}{B}(R\sin\theta)$$
$$+ C_2 \sin\frac{\pi}{B}(R\sin\theta)\Big] e^{-i\left(\frac{k}{k_o}\right)R} \, dR\, d\theta$$
$$+ 8\frac{B}{A} i \int_{\theta=\theta_o}^{\frac{\pi}{2}} \int_{R=0}^{\frac{B}{\sin\theta}} (A - R\cos\theta) \Big[C_1 (B - R\sin\theta) \cos\frac{\pi}{B}(R\sin\theta)$$
$$+ C_2 \sin\frac{\pi}{B}(R\sin\theta)\Big] e^{-i\left(\frac{k}{k_o}\right)R} \, dR\, d\theta$$

where

(C-7) $$\tan\theta_o = \frac{B}{A} \, .$$

This change of variables is helpful because the singular point at $\eta = \xi = 0$ in the integrand of (C-1) is troublesome for computer evaluation. The region of integration in (C-6) in the R-θ plane is shown shaded in Fig. C-1.

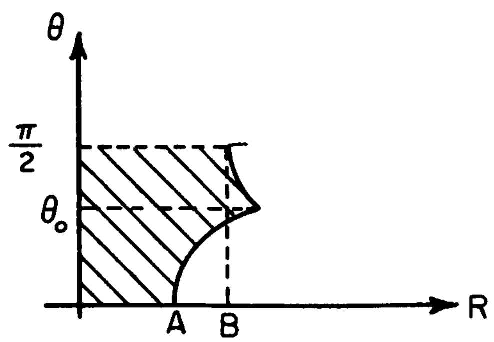

Fig. C-1. Region of integration in R-θ plane.

Next let

(C-8) $$\frac{k}{k_o} = C e^{-i\phi} .$$

Then

(C-9) $$e^{-i\left(\frac{k}{k_o}\right)R} = [\cos(RC\cos\phi) - i\sin(RC\cos\phi)]e^{-RC\sin\phi} .$$

C_1 and C_2 may be split into their real and imaginary parts:

(C-10) $$C_1 = C_{1r} - iC_{1i} = \frac{1}{4\pi B^2}\left[C^2\cos 2\phi - iC^2\sin 2\phi - \left(\frac{\pi}{B}\right)^2\right]$$

(C-11) $$C_2 = C_{2r} - iC_{2i} = \frac{1}{4\pi^2 B}\left[C^2\cos 2\phi - iC^2\sin 2\phi + \left(\frac{\pi}{B}\right)^2\right] .$$

Then the normalized conductance G_n and normalized susceptance B_n, i.e.,

(C-12) $$Y_n = G_n + iB_n,$$

may be written

(C-13) $$G_n = 8\frac{B}{A}\int_{\theta=0}^{\theta_o}\int_{R=0}^{\frac{A}{\cos\theta}}\{[C_{1r}\sin(RC\cos\phi) + C_{1i}\cos(RC\cos\phi)]f(R,\theta)$$

$$+ [C_{2r}\sin(RC\cos\phi)+C_{2i}(RC\cos\phi)]g(R,\theta)\}e^{-RC\sin\phi}dR\,d\phi$$

$$+ 8\frac{B}{A}\int_{\theta=\theta_o}^{\frac{\pi}{2}}\int_{R=0}^{\frac{B}{\sin\theta}}\{[C_{1r}\sin(RC\cos\phi)+C_{1i}\cos(RC\cos\phi)]f(R,\theta)$$

$$+ [C_{2r}\sin(RC\cos\phi)+C_{2i}\cos(RC\cos\phi)]g(R,\theta)\}e^{-RC\sin\phi}dR\,d\phi$$

(C-14) $$B_n = 8\,\frac{B}{A}\int_{\theta=0}^{\theta_o}\int_{R=0}^{\frac{A}{\cos\theta}}\{[C_{1r}\cos(RC\cos\phi)-C_{1i}\sin(RC\cos\phi)]f(R,\theta)$$

$$+[C_{2r}\cos(RC\cos\phi)-C_{2i}\sin(RC\cos\phi)]g(R,\theta)\}e^{-RC\sin\phi}dR\,d\theta$$

$$+\,8\,\frac{B}{A}\int_{\theta=\theta_o}^{\frac{\pi}{2}}\int_{R=0}^{\frac{B}{\sin\theta}}\{[C_{1r}\cos(RC\cos\phi)-C_{1i}\sin(RC\cos\phi)]f(R,\theta)$$

$$+[C_{2r}\cos(RC\cos\phi)-C_{2i}\sin(RC\cos\phi)]g(R,\theta)\}e^{-RC\sin\phi}dR\,d\theta$$

where

(C-15) $$f(R,\theta) = (A-R\cos\theta)(B-R\sin\theta)\cos\frac{\pi}{B}(R\sin\theta)$$

(C-16) $$g(R,\theta) = (A-R\cos\theta)8\sin\frac{\pi}{B}(R\sin\theta)\ .$$

These integrals have been evaluated numerically on the IBM 1620 Digital computer at Ohio State. The procedure used is as follows.

Each double integral is evaluated as an iterated integral, the integration on R being done first. Simpson's rule is used throughout. First, with θ held constant at each of the values 0, 0.1 $(\pi/2)$, 0.2$(\pi/2)$, $\cdots$, $(\pi/2)$, the R-integral is computed by breaking the range of R into ten subintervals, evaluating the integrand at the end-points of the subintervals, and summing according to Simpson's rule. These values, which form the integrand for the θ-integral, are then summed again by Simpson's rule to evaluate the θ-integral.

The program for these calculations was written in Fortran (OSU Version 2) and is included below.

```
C     ADMITTANCE OF RECTANGULAR APERTURE
      READ 1, N
      DO 14 M = 1, N, 1
      READ F,A,B,C,PHI
      D=PHI/57.3
      Z = ATAN(B/A)
      DIMENSION BR(2), BI(2), TH(11), ARG(4)
      BR(1)=(C*C*COS(2.*D)-9.8697/B/B)/12.6464/B/B
      BR(2)=(C*C*COS(2.*D)+9.8697/B/B)/39.7299/B
      BI(1)=C*C*SIN(2.*D)/12.6464/B/B
      BI(2)=C*C*SIN(2.*D)/39.7299/B
      DO 13 J=1,11,1
      P=J-1
      TH(J)=0.157079*P
      IF(Z-TH(J)) 1,2,2
 1    DR=B/10./SIN(TH(J))
      GO TO 3
 2    DR=A/10./COS(TH(J))
 3    DO 8 I=1,11,1
      H=I-1
      R=H*DR
      ER=EXP(-R*C*SIN(D))
      F=(A-R*COS(TH(J)))*(B-R*SIN(TH(J)))*COS(3.1416/B*R*SIN(TH(J)))
      G=(A-R*COS(TH(J)))*SIN(3.1416/B*R*SIN(TH(J)))
      ARG(1)=(BR(1)*SIN(R*C*COS(D))+BI(1)*COS(R*C*COS(D)))*F*ER
      ARG(2)=(BR(2)*SIN(R*C*COS(D))+BI(2)*COS(R*C*COS(D)))*G*ER
      ARG(3)=(BR(1)*COS(R*C*COS(D))-BI(1)*SIN(R*C*COS(D)))*F*ER
      ARG(4)=(BR(2)*COS(R*C*COS(D))-BI(2)*SIN(R*C*COS(D)))*G*ER
      GO TO (4,6,7,6,7,6,7,6,7,6,5),I
 4    S=0.
      T=0.
 5    S=S+2.6667*B/A*(ARG(1)+ARG(2))*DR
      T=T+2.6667*B/A*(ARG(3)+ARG(4))*DR
      GO TO 8
 6    S=S+10.6667*B/A*(ARG(1)+ARG(2))*DR
      T=T+10.6667*B/A*(ARG(3)+ARG(4))*DR
      GO TO 8
 7    S=S+5.3333*B/A*(ARG(1)+ARG(2))*DR
      T=T+5.3333*B/A*(ARG(3)+ARG(4))*DR
 8    CONTINUE
      GO TO (9,11,12,11,12,11,12,11,12,11,10),J
 9    GN=0.
      BN=0.
 10   GN=GN+0.05236*S
      BN=BN+0.05236*T
      GO TO 13
 11   GN=GN+0.20944*S
      BN=BN+0.20944*T
      GO TO 13
 12   GN=GN+0.10472*S
      BN=BN+0.10472*T
 13   CONTINUE
      YNM=SQRT(GN*GN+BN*BN)
      ANGY=57.3*ATAN(BN/GN)
      X=M
 14   PUNCH F,X,A,B,C,PHI,GN,BN,YNM,ANGY
      STOP
      END
```